Resolution of Unsteady Navier-stokes Equations with the $C_{a,b}$ Boundary condition

Jaouad EL-Mekkaoui, Ahmed Elkhalfi
Mechanical engineering laboratory
Faculty of sciences and techniques-B.P.
2202 Route Imouzzer Fes

Abdeslam Elakkad
Department of mathematics
Regional Centre for Professions of Education and Training,
Fes, B.P: 243 Sefrou Morocco

***Abstract*—in this work, we introduce the unsteady incompressible Navier-Stokes equations with a new boundary condition, generalizes the Dirichlet and the Neumann conditions. Then we derive an adequate variational formulation of time-dependent Navier- Stokes equations. We then prove the existence theorem and a uniqueness result. A Mixed finite-element discretization is used to generate the nonlinear system corresponding to the Navier-Stokes equations. The solution of the linearized system is carried out using the GMRES method. In order to evaluate the performance of the method, the numerical results are compared with others coming from commercial code like Adina system.**

Keywords—Unsteady Navier-Stokes Equations; Mixed Finite Element Method; $C_{a,b,c}$ boundary condition; Adina system.

I. Introduction

The two firsts PDEs given in section 2 constitute the basis for computational modeling of the flow of an in compressible Newtonian fluid. For the equations, we offer a choice of two-dimensional domains on which the problem can be posed, along with boundary conditions and other aspects of the problem, and a choice of finite element discretizations on a quadrilateral element mesh, whereas the discrete Navier-Stokes equations require a method such as the generalized minimum residual method (GMRES), which is designed for non symmetric systems [9, 19]. The key for fast solution lies in the choice of effective preconditioning strategies. The package offers a range of options, including algebraic methods such as incomplete LU factorizations, as well as more sophisticated and state- of-the-art multigrid methods designed to take advantage of the structure of the discrete linearized Navier-Stokes equations. In addition, there is a choice of iterative strategies, Picard iteration or Newton method, for solving the nonlinear algebraic systems arising from the latter problem.

This paper presents the unsteady Navier-Stocks equations with a new boundary condition noted by $C_{a,b}$. This condition generalizes the known conditions, especially the conditions of Dirichlet, Neumann...

If a *and* b are the real numbers strictly positive such that $a << b,$ then $C_{a,b}$ is the Neumann boundary condition and if $a >> b$ then $C_{a,b}$ is the Dirichlet boundary condition. For that a is called the Dirichlet coefficient and b is the Neumann coefficient. So, we prove that the weak formulation of the proposed modeling has a unique solution. To calculate this latter, we use the discretization by mixed finite element method. Moreover, to compare our solution with the some previously ones, as ADINA system, some numerical results are shown.

The paper is organized as follows. Section II presents the model problem. In the section III we show the existence and uniqueness of the solution of the standard weak formulation and of semi-descretization.

In section IV we describe the approximation of the standard weak formulation using mixed finite elements and Picards nonlinear iteration and using GMRES algorithm to solve it. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in section V.

II. Unsteady Incompressible Navier-Stokes Equations

We consider the unsteady Navier-Stokes equations for the flow with constant viscosity.

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu\nabla^2\vec{u} + \vec{u}.\nabla\vec{u} + \nabla p = \vec{f} \ \text{ in } \ \Omega, \\ \nabla.\vec{u} = 0 \ \text{ in } \quad \Omega, \\ \vec{u}(0) = \vec{u}_0 \ \text{ in } \quad \Omega. \end{cases} \tag{1}$$

Where $\nu \succ 0$ a given constant is called the kinematic viscosity. $\vec{u}$ is the fluid velocity, p is the pressure field, ∇ is the gradient and $\nabla.$ is the divergence operator.

The boundary value problem that is considered is posed on two or three-dimensional domain Ω, is defined as:

$$C_{a,b} \ : \quad a\vec{u} + b\left(\nu\nabla\vec{u} - pI\right)\vec{n} = \vec{t} \ in \ \Gamma := \partial\Omega \tag{2}$$

Where $\vec{n}$ that is the usual outward-pointing normal boundary.

$\vec{t} \in H^{\frac{1}{2}}(\Gamma)$, a and b are the function nonzero continuous defined on Γ verify:
There are two strictly positive constants α_1 and β_1, such that:

$$\alpha_1 \le \frac{a(x)}{b(x)} \le \beta_1 \text{ for all } x \in \Gamma \tag{3}$$

This system is the basis for computational modeling of the flow of an incompressible fluid such as air or water. The presence of the nonlinear convection term $\vec{u}\nabla\vec{u}$ means that boundary value problems associated with the Navier-Stokes equations can have more than one solution.

We define the spaces

$$h^1(\Omega) = \left\{ u : \Omega \to IR / u; \frac{\partial u}{\partial x}; \frac{\partial u}{\partial y} \in L^2(\Omega) \right\} \tag{4}$$

$$H^1(\Omega) = \left[h^1(\Omega)\right]^2 \tag{5}$$

$$H_N^1(\Omega) = \left\{ \vec{v} \in H^1(\Omega) / \vec{v}.\vec{n} = 0 \text{ on } \Gamma \right\} \tag{6}$$

$$\xi = \left\{ \vec{v} \in D(\overline{\Omega}) / \nabla.\vec{v} \text{ and } \vec{v}.\vec{n} = 0 \text{ on } \Gamma \right\} \tag{7}$$

$$V(\Omega) = \left\{ \vec{v} \in H_N^1(\Omega) / \nabla.\vec{v} = 0 \text{ on } \Gamma \right\} \tag{8}$$

$$H = \left\{ \vec{v} \in L^2(\Omega) / \nabla.\vec{v} \text{ and } \vec{v}.\vec{n} = 0 \text{ on } \Gamma \right\} \tag{9}$$

$L^p(0,T;X)$ is the space of strongly measurable functions $f:(0,T) \to X$ such that:

$$\| f \|_{L^p(0,T;X)} = \left[\int_0^T \| f(t) \|_X^2 \, dt \right]^{\frac{1}{2}} \prec \infty,$$

$$\| f \|_{L^\infty(0,T;X)} = \sup \text{ess}_{t \in (0,T)} \| f(t) \|_X \prec \infty.$$

For $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $H^1(\Omega)$ we define, as usual, the following forms:

$$a_0(\vec{u},\vec{v}) = \nu \int_\Omega \nabla\vec{u} : \nabla\vec{v} + \int_\Gamma \frac{a}{b} \vec{u}\vec{v}$$

$$b(\vec{u},q) = -\int_\Omega q \nabla.\vec{u}$$

$$d(p,q) = \int_\Omega p.q$$

$$a_1(\vec{z},\vec{u},\vec{v}) = \int_\Omega (\vec{z}\nabla\vec{u})\vec{v}$$

$$c(\vec{z},\vec{u},\vec{v}) = a_0(\vec{u},\vec{v}) + a_1(\vec{z},\vec{u},\vec{v})$$

$$l(\vec{v}) = \int_\Gamma \frac{a}{b} \vec{t}.\vec{v} + \int_\Omega \vec{f}.\vec{v}.$$

For $\vec{u}$ and $\vec{w}$ in $L^2(0,T;V)$ we consider the function $t \to A_0\vec{u}(t)$ defined a.e. on $[0, T]$ by:

$$A_0\vec{u}(t) \in V', \quad \langle A_0\vec{u}(t), \vec{v} \rangle = a_0(\vec{u}(t), \vec{v}) \quad \forall \vec{v} \in V.$$

It can be readily checked that $t \to A_0\vec{u}(t) \in L^2(0,T;V').$
Next, consider the mapping $t \to A_1(\vec{w}(t), \vec{u}(t))$ defined a.e On $[0, T]$ by:

$$\begin{cases} A_1(\vec{w}(t),\vec{u}(t)) \in V', \\ \langle A_1(\vec{w}(t),\vec{u}(t)), \vec{v} \rangle = a_1(\vec{w}(t), \vec{u}(t), \vec{v}) \quad \forall \vec{v} \in V^{\cdot} \end{cases}$$

for all $\vec{v} \in V.$

The standard weak formulation of (1), find $\vec{u} \in H_N^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega \frac{\partial \vec{u}}{\partial t} \vec{v} + c(\vec{z};\vec{u},\vec{v}) + b(\vec{v},q) = \int_\Omega \vec{f}\vec{v} + \int_\Gamma \frac{a}{b} \vec{t}\vec{v}, \tag{10}$$

$$b(\vec{u},q) = 0, \tag{11}$$

for all $\vec{v} \in H_N^1(\Omega)$ *and* $q \in L^2(\Omega).$

The spaces $H^1(\Omega)$, $H_N^1(\Omega)$ and V equipped with the norms

$$\|\vec{v}\|_{J,\Omega} = \left(\nu \int_\Omega \nabla\vec{v} : \nabla\vec{v} d\Omega + \int_\Gamma \frac{a}{b} \vec{v}\vec{v} d\gamma \right)^{\frac{1}{2}}$$

$$\|\vec{v}\|_{1,\Omega} = \left(|\vec{v}|_{1,\Omega}^2 + \|\vec{v}\|_{0,\Omega}^2 \right)^{\frac{1}{2}}$$

The spaces H and $L^2(\Omega)$ equipped with the norm

$$\|\vec{v}\|_{0,\Omega} = (\vec{v},\vec{v})^{\frac{1}{2}}$$

$C_{a,b}$ is new boundary condition, for that we need to show the existence and uniqueness theorems for this modeling.

III. Existence And Uniquness Of The Solution

A. Weak formulation

THEOREM 1. There are two strictly positive constants c_1 and c_2 such that:

$$c_1 \|\vec{v}\|_{1,\Omega} \le \|\vec{v}\|_{J,\Omega} \le c_2 \|\vec{v}\|_{1,\Omega} \quad \textit{for all } \vec{v} \in H^1(\Omega) \tag{12}$$

PROOF. 1) The mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is continuous

(See [6] theorem 1, 2), then there exists $c \succ 0$ such that: $\|\vec{v}\|_{0,\Gamma} \le c\|\vec{v}\|_{1,\Omega}$ for all $\vec{v} \in H^1(\Omega)$. Using this result and the result (3) give,

$$\|\vec{v}\|_{J,\Omega} \le c_2 \|\vec{v}\|_{1,\Omega} \textit{ for all } \vec{v} \in H^1(\Omega),$$

with $c_2 = \left(\beta_1 c^2 + \nu\right)^{\frac{1}{2}}$. On other hand, according to 5.55 in [1], there exists a constant $\rho \succ 0$ such that

$\|\vec{v}\|_{0,\Omega}^2 \le \rho(\|\nabla\vec{v}\|_{0,\Omega}^2 + \|\vec{v}\|_{0,\Gamma}^2), \ \forall \vec{v} \in H^1(\Omega)$, using (3) and

$\|\nabla\vec{v}\|_{0,\Omega}^2 \le \|\vec{v}\|_{J,\Omega}^2$, give

$c_1\|\vec{v}\|_{1,\Omega} \le \|\vec{v}\|_{J,\Omega}$ *for all* $\vec{v} \in H^1(\Omega)$,

with $c_1 = \left(\dfrac{\rho C}{\nu\alpha_1} + \dfrac{1}{\nu}\right)^{-\frac{1}{2}}$ and $C = \max\{\alpha_1; \nu\}$.

Finally $c_1\|\vec{v}\|_{1,\Omega} \le \|\vec{v}\|_{J,\Omega} \le c_2\|\vec{v}\|_{1,\Omega}$ for all $\vec{v} \in H^1(\Omega)$.

- $(H^1(\Omega), \|\,.\|_{1,\Omega})$ is a real space and $H_N^1(\Omega)$ is closed in $H^1(\Omega)$ and $\|\,.\|_{1,\Omega}$ and $\|\,.\|_{J,\Omega}$ are equivalent norms, then $(H^1(\Omega), \|\,.\|_{J,\Omega})$ and $(H_N^1(\Omega), \|\,.\|_{J,\Omega})$ are a reals Hilbert spaces.

LEMMA 2. *All* $\vec{v} \in H_0^1(\Omega)$ satisfy

$$\|\vec{v}\|_{0,4,\Omega} \le 2^{\frac{1}{4}} |\vec{v}|_{1,\Omega}^{\frac{1}{2}} \|\vec{v}\|_{0,\Omega}^{\frac{1}{2}}. \tag{13}$$

PROOF. See lemma 1.5 chapter V [2].

LEMMA 3. *All* $\vec{v} \in H^1(\Omega)$ satisfy

$$\|\vec{v}\|_{0,4,\Omega} \le 2^{\frac{1}{4}} |\vec{v}|_{1,\Omega}^{\frac{1}{2}} \|\vec{v}\|_{0,\Omega}^{\frac{1}{2}}. \tag{14}$$

PROOF. Ω is a bounded open of IR^2 with Lipschitz continuous boundary, then there exists a sequence $(\Omega_n)_{n\ge 0}$ of opens, such that

For all $n \ge 0, \Omega_n \subset \Omega$ *and* $\lim_{n\to\infty} \Omega_n = \Omega$.

Let φ_n be a function defined as:

$$\begin{cases} \varphi_n = 1 \ \ on \ \ \Omega_n, \\ \varphi_n = 1 \ \ on \ \ \Omega - \Omega_n. \end{cases} \tag{15}$$

Then for all $\vec{v} \in H^1(\Omega)$, we get $\varphi_n\vec{v} \in H_0^1(\Omega)$, for all $n \ge 0$, According to lemma 2,

$$\|\varphi_n\vec{v}\|_{0,4,\Omega} \le 2^{\frac{1}{4}} |\varphi_n\vec{v}|_{1,\Omega}^{\frac{1}{2}} \|\varphi_n\vec{v}\|_{0,\Omega}^{\frac{1}{2}}.$$

By applying dominated convergence Theorem, we get

$$\|\vec{v}\|_{0,4,\Omega} \le 2^{\frac{1}{4}} |\vec{v}|_{1,\Omega}^{\frac{1}{2}} \|\vec{v}\|_{0,\Omega}^{\frac{1}{2}}.$$

LEMMA 4. when $\vec{w}$ and $\vec{u}$ belong both to $L^2(0,T;V) \cap L^\infty(0,T;H)$ then

$$A_1(\vec{w},\vec{u}) \in L^2(0,T;V') \tag{16}$$

PROOF. $\vec{w},\ \vec{u},\ \vec{v} \in V$, we have

$$a_1(\vec{w},\vec{u},\vec{v}) + a_1(\vec{w},\vec{v},\vec{u}) = \int_\Omega \vec{w}.(\nabla\vec{u}.\vec{v} + \nabla\vec{v}.\vec{u}) = \int_\Omega \vec{w}.\nabla(\vec{u}.\vec{v}).$$

Use Green's theorem to show that

$$a_1(\vec{w},\vec{u},\vec{v}) + a_1(\vec{w},\vec{v},\vec{u}) = \int_{\partial\Omega} (\vec{z}.\vec{n})(\vec{u}.\vec{v}) - \int_\Omega div\ \vec{z}.(\vec{u}.\vec{v}) = 0.$$

Then, $a_1(\vec{w},\vec{u},\vec{v}) = -a_1(\vec{w},\vec{v},\vec{u})$, and we have the upper bound:

$$a_1(\vec{w},\vec{u},\vec{v}) \le c_1\|\vec{w}\|_{0,4,\Omega}\|\vec{u}\|_{0,4,\Omega}|\vec{v}|_{1,\Omega}$$

Therefore

$$\|A_1(\vec{w}(t),\vec{u}(t))\|_* \le c_1\|\vec{w}(t)\|_{0,4,\Omega}\|\vec{u}(t)\|_{0,4,\Omega}. \tag{17}$$

We make use of lemma 2. According to (14) and (17), we have

$$\|A_1(\vec{w}(t),\vec{u}(t))\|_*^2 \le 2c_1|\vec{w}(t)|_{1,\Omega}|\vec{u}(t)|_{1,\Omega}\|\vec{w}(t)\|_{0,\Omega}\|\vec{u}(t)\|_{0,\Omega} \tag{18}$$

Therefore,

$$\int_0^T \|A_1(\vec{w}(t),\vec{u}(t))\|_*^2 dt \le C\|\vec{w}\|_{L^\infty(0,T;H)}\|\vec{u}\|_{L^\infty(0,T;H)}\|\vec{w}\|_{L^2(0,T;V)}\|\vec{u}\|_{L^2(0,T;V)}$$

This proves (16).

In the course of this proof we have shown that, if $\vec{u}$ belongs both to $L^2(0,T;V) \cap L^\infty(0,T;H)$ then

$$\|A_1(\vec{u},\vec{u})\|_{L^2(0,T;V')} \le C\|\vec{u}\|_{L^\infty(0,T;H)}\|\vec{u}\|_{L^2(0,T;V)} \tag{19}$$

With these spaces and forms, consider the following variational formulation of problem (1)-(2).

For a given function $\vec{f} \in L^2(0,T;H^{-1}(\Omega))$ and a given element $\vec{u}_0$ of H, find $\vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ Such that

$$\begin{cases} \bullet\ \dfrac{d}{dt}(\vec{u}(t),\vec{v}) + c(\vec{u}(t);\vec{u}(t),\vec{v}) = \int_\Omega \vec{f}\vec{v} + \int_\Gamma \vec{t}\vec{v}, \\ \quad \forall\vec{v} \in V, \ \text{in} \ \ D'(]0,T[), \\ \bullet\ \vec{u}(0) = \vec{u}_0 \quad \text{in} \ \ \Omega. \end{cases} \tag{20}$$

THEOREM 5. Let $\vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ be solution of (20). Then $\dfrac{d\vec{u}}{dt} \in L^2(0,T;V')$.

PROOF. We have

$$c(\vec{u}(t);\vec{u}(t),\vec{v}) = \langle A_0\ \vec{u}(t) + A_1(\vec{u}(t),\vec{u}(t)),\vec{v}\rangle$$

for all $\vec{v} \in V$.

Therefore, by Lemma 1.1 chapter V in [2], each solution $\vec{u}$ of (20) satisfies in D'(]0, T[):

$$\left\langle \frac{d\vec{u}}{dt}(t),\vec{v}\right\rangle = \int_\Omega \vec{f}\vec{v} + \int_\Gamma \frac{a}{b}\vec{t}\vec{v} - \langle A_0\ \vec{u}(t) + A_1(\vec{u}(t),\vec{u}(t)),\vec{v}\rangle$$

for all $\vec{v} \in V$.

Now, by hypothesis

$$l : \vec{v} \to \int_\Omega \vec{f}\vec{v} + \int_\Gamma \frac{a}{b}\vec{t}\vec{v} \in L^2(0,T;V')$$

and we have mentioned previously that $A_0\vec{u} \in L^2(0,T;V')$ when $\vec{u} \in L^2(0,T;V)$. Furthermore $A_1(\vec{u},\vec{u}) \in L^2(0,T;V')$

Hence, $\frac{d\vec{u}}{dt} \in L^2(0,T;V')$.

According to Theorem 5 and Theorem 1.1 in [2] chapter V, $\vec{u} \in C^0([0,T];H)$.

In this case, it is perfectly allowable to prescribe the value of $\vec{u}$ at *t=0*. Moreover, it stems from the above proof that problem (20) can be equivalently stated as follows:

For a given function $\vec{f} \in L^2(0,T;H^{-1}(\Omega))$ and a given element $\vec{u}_0$ of H, find

$$\begin{cases} \bullet\, \vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \\ with\ \frac{d\vec{u}}{dt} \in L^2(0,T;V')\ such\, that \\ \bullet\, \frac{d\vec{u}}{dt} + A_0\,\vec{u} + A_1(\vec{u},\vec{u}) = L, \\ \bullet\, \vec{u}(0) = \vec{u}_0 \quad \text{in } \Omega. \end{cases} \tag{21}$$

We consider the following problem:

For a given function $\vec{f} \in L^2(0,T;H^{-1}(\Omega))$ and a given element $\vec{u}_0$ of H, find $(\vec{u},p)$ such that

$$\begin{cases} \bullet\, \vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \\ with\ \frac{d\vec{u}}{dt} \in L^2(0,T;V')\ and\ D'(\Omega\times]0,T[) \\ \bullet\, \frac{\partial\vec{u}}{\partial t} - \nu\nabla^2\vec{u} + \vec{u}.\nabla\vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega, \\ \bullet\, \vec{u}(0) = \vec{u}_0 \quad \text{in } \Omega. \end{cases} \tag{22}$$

THEOREM 6. Problem (21) and (22) are equivalent.

To show this Theorem we need the following lemmas

LEMMA7. Let $\vec{g} \in H^{\frac{1}{2}}(\Gamma)$ satisfies $\int_\Gamma \vec{g}\vec{n}d\lambda = 0$. Then there exists a function $\vec{u} \in H^1(\Omega)$ such that

$$div\,\vec{u} = 0\ and\ \vec{u} = \vec{g}\ on\ \Gamma.$$

PROOF. See Temam [20].

LEMMA8**.** The divergence operator is an isomorphism from $V^\perp$ onto $L_0^2(\Omega)$, verifies: There exists $\alpha \succ 0$ such that

$$\|div\,\vec{v}\|_{0,\Omega}^2 \ge \alpha\|\vec{v}\|_{J,\Omega}^2 \ \ \forall\vec{v} \in V^\perp.$$

PROOF. Let $\vec{v} \in H_N^1(\Omega)$ By Green's formula:

$$\int_\Omega div\,\vec{v}\,dx = \int_\Gamma \vec{v}.\vec{n}\,d\lambda = 0.$$

Let q a function of $L_0^2(\Omega)$. As Ω is bounded, there exists some function $\theta \in L^2(\Omega)$ such that

$$\Delta\theta = q\ in\ \Omega.$$

We set $\vec{v}_1 = \overrightarrow{grad}\,\theta\ \ in\ \ \Omega$. Then $div\,\vec{v}_1 = \Delta\theta = q\ \ in\ \ \Omega$; moreover, by Green's formula $\int_\Gamma \vec{v}_1.\vec{n}\,d\lambda = \int_\Omega div\,\vec{v}_1\,dx = \int_\Omega q\,dx = 0.$

Also $\gamma_0\vec{v}_1 \in H^{\frac{1}{2}}(\Gamma)$. Therefore, we can apply Lemma 7: there exists $\vec{w}_1 \in H^1(\Omega)$ such that $div\,\vec{w}_1 = 0$ and $\gamma_0\vec{v}_1 = \gamma_0\vec{w}_1$. then $\vec{v} = \vec{v}_1 - \vec{w}_1$ is the required function since $\vec{v} \in H_0^1(\Omega)$ and $div\,\vec{v} = q$.

Finally, it follows from the open mapping Theorem (cf.Yosida [21]) that the inverse of div is continuous from $L_0^2(\Omega)$ onto $V^\perp$, therefore div is an isomorphism.

LEMMA 9. Let $\vec{l} \in H^{-1}(\Omega)$ and satisfy

$$\langle \vec{l},\vec{v} \rangle = 0,\ \forall\vec{v} \in V^\perp$$

Then there exists exactly one function φ in $L_0^2(\Omega)$ such that:

$$\langle \vec{l},\vec{v} \rangle = \int_\Omega \varphi div\vec{v}dx = -\langle \overrightarrow{grad}\varphi,\vec{v} \rangle. \tag{23}$$

$\forall\vec{v} \in H_N^1(\Omega).$

PROOF. Consider the following problem:

Find $\vec{u} \in V^\perp$ satisfying

$$\langle div\,\vec{u}, div\,\vec{v} \rangle = \langle \vec{l},\vec{v} \rangle, \forall\vec{v} \in V^\perp. \tag{24}$$

By the Lemma 8 and the Lax-Milgram's Theorem, (24) has a unique solution $\vec{u}$ in $V^\perp$. Then,

$$\langle div\,\vec{u}, div\,\vec{v} \rangle = \langle \vec{l},\vec{v} \rangle, \forall\vec{v} \in H_N^1(\Omega).$$

We set $\varphi = div\,\vec{u} \in L_0^2(\Omega)$ and we find (23).

It remains to prove that φ is unique in $L_0^2(\Omega)$. But clearly, if $\varphi \in L_0^2(\Omega)$ and $\langle \varphi, div\,\vec{v} \rangle = 0\ \ \forall\vec{v} \in H_N^1(\Omega)$, then $\varphi = 0$ since div maps $H_N^1(\Omega)$, onto $L_0^2(\Omega)$.

PROOF OF THEOREM 6. Clearly, if $(\vec{u},p)$ is a solution of (22), then $\vec{u}$ satisfies (21).

Conversely, let $\vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$

A solution of (21) and consider the mapping defined on $H_N^1(\Omega)$, by:

$$L(\vec{v},t) = \int_0^t \left\{ \langle \vec{f}(s),\vec{v} \rangle - c(\vec{u}(t);\vec{u}(t),\vec{v}) \right\} ds - (\vec{u}(t),\vec{v}) + (\vec{u}_0,\vec{v})$$

For each *t*, L is a linear functional on $H_N^1(\Omega)$, that vanishes on *V*. Hence, according to Lemma 9, for each *t* there exists exactly one function $P(t) \in L_0^2(\Omega)$, such that

$$L(\vec{v},t) = -\left\langle \overrightarrow{grad}\, P(t), \vec{v} \right\rangle \; \forall \vec{v} \in H_N^1(\Omega).$$

In other words,

$$(P(t),\vec{v}) = \int_0^t \left\{ \left\langle \vec{f}(s),\vec{v} \right\rangle - c(\vec{u}(s);\vec{u}(s),\vec{v}) \right\} ds - (\vec{u}(t),\vec{v}) + (\vec{u}_0,\vec{v}) \quad \forall \vec{v} \in H_N^1(\Omega). \tag{25}$$

By using Lemma 4, it can be checked that

$P \in C^0\left([0,T]; L_0^2(\Omega)\right)$

Next, by differentiating (25), we get:

$$\left(\frac{dP(t)}{dt},\vec{v}\right) = \left\{ \left\langle \vec{f}(s),\vec{v} \right\rangle - c(\vec{u}(s);\vec{u}(s),\vec{v}) \right\} - \left(\frac{d\vec{u}(t)}{dt},\vec{v}\right) + (\vec{u}_0,\vec{v}) \quad \forall \vec{v} \in H_N^1(\Omega). \tag{26}$$

Thus, if we set $p = \dfrac{dP}{dt}$ in $D'(\Omega \times]0,T[)$, we find the second equation of (22).

THEOREM 10. Problem (21) has a unique solution in $\vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$

PROOF. The same steps of proof of Theorem 1.5 chapter VI in [2] but the spaces *V*, *H*, $H_N^1(\Omega)$... are note the same. .

B. Semi-discritisation

In this paragraph, we propose to analyze a very simple one-step method in order to illustrate the type of argument that is often used when dealing with semi-discretization. Consider again the problem (20).

find $\vec{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ Such that

$$\begin{cases} \bullet \dfrac{d}{dt}\left(\vec{u}(t),\vec{v}\right) + c(\vec{u}(t);\vec{u}(t),\vec{v}) = l(\vec{v}), \\ \quad \forall \vec{v} \in V, \text{ in } D'(]0,T[), \\ \bullet \vec{u}(0) = \vec{u}_0 \quad \text{in } \Omega. \end{cases} \tag{27}$$

Let $k = \dfrac{T}{N}$ and t_n the subdivisions of [0, T]: $t_n = nk$; $0 \le n \le N$.

Now, suppose that an approximation, $\vec{u}^n \in V$, of $\vec{u}(t_n)$ is available and consider the following problem:

$$\begin{cases} \bullet \; find\ \vec{u}^{n+1} \in V\ such\,that \\ \bullet \dfrac{1}{k}\left(\vec{u}^{n+1} - \vec{u}^n, \vec{v}\right) + c(\vec{u}^n;\vec{u}^{n+1},\vec{v}) \\ \quad = \int_\Omega \vec{f}(t_{n+1})\vec{v} + \int_\Gamma \vec{t}(t_{n+1})\vec{v}, \forall \vec{v} \in V. \end{cases} \tag{28}$$

$\vec{u}^n$, $\vec{f}(t_{n+1})$ and $\vec{t}(t_{n+1})$ are given respectively in V and V', it follows that (28) can be expressed in the from:

$$\begin{cases} \bullet \; find\ \vec{u}^{n+1} \in V\ such\,that \\ \bullet \left(\vec{u}^{n+1}, \vec{v}\right) + kc(\vec{u}^n;\vec{u}^{n+1},\vec{v}) \\ = k\left[\int_\Omega \vec{f}(t_{n+1})\vec{v} + \int_\Gamma \vec{t}(t_{n+1})\vec{v},\right] + (\vec{u}_n,\vec{v}) \;\forall \vec{v} \in V. \end{cases} \tag{29}$$

Thus, we are asked to solve a linear boundary value problem associated with the bilinear form:

$$(\vec{u},\vec{v}) \to (\vec{u},\vec{v}) + kc(\vec{u}^n;\vec{u},\vec{v}).$$

This form is continuous in $V \times V$ and V-elliptic since

$$(\vec{v},\vec{v}) + kc(\vec{u}^n;\vec{v},\vec{v}) = \|\vec{v}\|_{0,\Omega}^2 + k\|\vec{v}\|_{J,\Omega}^2$$

Therefore, by Lax-Mailgram's theorem [2], problem (29) has a unique solution $\vec{u}^{n+1}$ in *V*.

IV. DISCRETIZATION BY MIXED FINITE ELEMENTS

Our goal here is to consider the unsteady Navier-Stokes equations with $C_{a,b}$ boundary conditions in a two dimensional domain and to approximate them by a mixed finite element method.

Mixed finite element discretization of the weak formulation of the Navier- Stokes equations gives rise to a nonlinear system of algebraic equations. Two classical iterative procedures for solving this system are Newton iteration and Picard iteration

Let T_h ; $h \succ 0$, be a family of rectangulations of Ω. For any $T \in T_h$.

We denote by h_T the diameter of a simplex, by h_E the diameter of a face E of T, and we set $h = \max_{T \in T_h}\{h_T\}$.

A discrete weak formulation is defined using finite dimensional spaces $X_h^1 \subset H_N^1(\Omega)$ and $M^h \subset L_0^2(\Omega)$

The discrete version of (10)-(11) is:

find $\vec{u}_h \in X_h^1$ and $p_h \in M^h$ such that:

$$\bullet \int_\Omega \frac{\partial \vec{u}_h}{\partial t}\vec{v}_h + \nu \int_\Omega \nabla \vec{u}_h : \nabla \vec{v}_h + \int_{\partial\Omega} \frac{a}{b}\vec{u}_h.\vec{v}_h + a_1(\vec{u}_h,\vec{u}_h,\vec{v}_h) + b(\vec{v}_h,p_h) = \int_\Omega \vec{f}\vec{v}_h + \int_\Gamma \frac{a}{b}\vec{t}\vec{v}_h \tag{30}$$

$$\bullet\, b(\vec{u}_h,q_h) = 0 \tag{31}$$

for all $\vec{v}_h \in X_h^1$ and $q_h \in M^h$.

We define the appropriate bases for the finite element spaces, leading to a non linear system of algebraic equations.

To define the corresponding linear algebra problem, we use a set of vector-valued basis functions $\{\vec{\varphi}_i\}_{i=1,\dots,n_u}$ So that

$$\vec{u}_h(t,x,y) = \sum_{j=1}^{n_u} u_j(t)\vec{\varphi}_j(x,y) \quad (32)$$

We introduce a set of pressure basis functions $\{\psi_k\}_{k=1,\dots,n_p}$ and set

$$p_h(t,x,y) = \sum_{k=1}^{n_p} p_k(t)\psi_k(x,y) \quad (33)$$

Where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.

$$D\frac{dU}{dt}(t) + [N(U(t)) + M]U(t) + BP(t) = L(t) \quad (34)$$

$$B^T U(t) = 0 \quad (35)$$

With

$$U(t) = (u_1(t), u_2(t), \dots, u_{n_u}(t))^T$$

$$P(t) = (p_1(t), p_2(t), \dots, p_{n_p}(t))^T$$

$$D(d_{i,j});\ d_{i,j} = \int_\Omega \vec{\varphi}_i.\vec{\varphi}_j,$$

$$N(U(t)) = (c_{i,j}),\ c_{i,j} = \sum_{k=1}^{n_u} u_k(t)\int_\Omega (\vec{\varphi}_j.\nabla\vec{\varphi}_k)\vec{\varphi}_i,$$

$$M = (m_{i,j}),\ m_{i,j} = \nu\int_\Omega \nabla\vec{\varphi}_j : \nabla\vec{\varphi}_i + \int_\Omega \frac{a}{b}\vec{\varphi}_j\vec{\varphi}_i,$$

$$M = (m_{i,j}),\ m_{i,j} = \nu\int_\Omega \nabla\vec{\varphi}_j : \nabla\vec{\varphi}_i + \int_\Omega \frac{a}{b}\vec{\varphi}_j\vec{\varphi}_i,$$

$$B = [b_{k,j}];\ b_{k,j} = -\int_\Omega \psi_k \nabla.\vec{\varphi}_j$$

$$L(t) = (l_i(t));\ l_i(t) = \int_\Omega \vec{f}(t).\vec{\varphi}_i + \int_\Gamma \frac{a}{b}\vec{t}(t)\vec{\varphi}_i$$

for $i, j = 1,\dots,n_u$, and $k = 1,\dots,n_p$.

Using the backward Euler method for the time derivative and substituting into (34)-(35), one obtains the following system of nonlinear equations in tensor notation:

$$U(0) = (u_1(0), u_2(0), \dots, u_{n_u}(0))^T, \quad (36)$$

$$D\frac{U^{n+1} - U^n}{k} + [N(U^{n+1}) + M]U^{n+1} + BP^{n+1} = L(t_{n+1}), \quad (37)$$

$$B^T U^{n+1} = 0. \quad (38)$$

With $u_1(0), u_2(0), \dots, u_{n_u}(0)$ are the coordinates in the basis $\{\vec{\varphi}_i\}_{i=1,\dots,n_u}$ of the approximation of $\vec{u}_0$ in X_h. Solution of the nonlinear system of equations, Eq. (34)-(35), can be carried out efficiently using Picards method, where we start with an initial guess $(U^{n,0}, P^{n,0}) \in IR^{n_u+n_p}$ and construct a sequence of iterates $(U^{n,m}, P^{n,m}) \in IR^{n_u+n_p}$ it converges to the solution of (34)-(35). In this approach we approximate the nonlinear convection term as follows:

$$[N(U^{n+1}) + M]U^{n+1} \cong [N(U^{n+1,m}) + M]U^{n+1,m+1}$$

For the finite-element basis functions, we chose to work with stable rectangular elements (Q2-Q1), where we use biquadratic approximation for the velocity components, bilinear approximation for the pressure, and stable triangular elements (P2-P1), where we use quadratic approximation for the velocity components and linear approximation for the pressure.

The linear system we need to solve within each iteration of Picards method has the following generic form:

$$\begin{pmatrix} A_0 + N & B_0^T \\ B_0 & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (39)$$

We use the generalized minimum residual method (GMRES) for solving the nonsymmetric systems. Preconditioning is a technique used to enhance the convergence of an iterative method to solve a large linear system iteratively. Instead of solving a system

$\Lambda x = b$, one solves a system $P^{-1}\Lambda x = P^{-1}b$, where P is the preconditioned. A good preconditioned should lead to fast convergence of the Krylov method. Furthermore, systems of the form Pz = r should be easy to solve. For the Navier-Stokes equations, the objective is to design a preconditioned that increases the convergence of an iterative method independent of the Reynolds number and number of grid points. We use a least-squares commutator preconditioning [10, 11,12].

V. NUMERICAL SIMULATIONS

In this section, some numerical results of calculations with mixed finite element method and ADINA system will be presented. Using our solver, we run two traditional test problems (driven cavity flow [9, 14, 15, 16, 17], Backward-facing step problem [10, 13]) and the flow over an obstacle [9] with a number of different model parameters.

EXAMPLE 1. Driven cavity flow. It is a model of the flow in a square cavity with the lid moving from left to right. Let the computational model: $\{y = 1; -1 \le x \le 1 / u_x = 1 - x^4\}$, a regularized cavity. The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

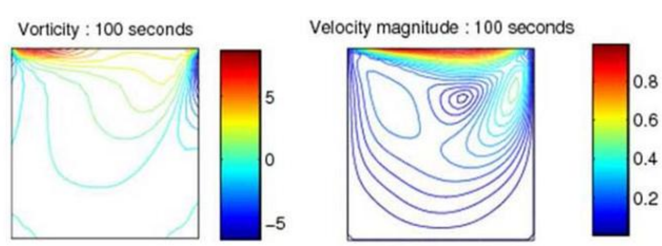

Fig.1. Vorticity (left plot), and velocity magnitude solution (right plot) using $P2-P1$ approximation a 64 × 64 square grid and Reynolds number Re =100.

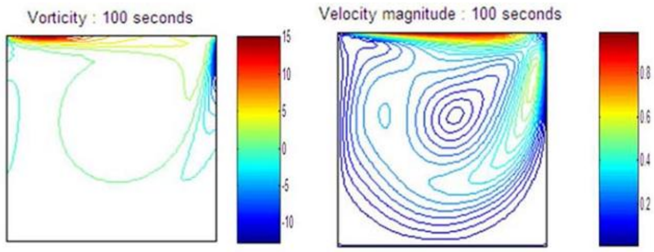

Fig.3. Vorticity (left plot), and velocity magnitude solution (right plot) using $P2 - P1$ approximation a 64 × 64 square grid and Reynolds number Re =400.

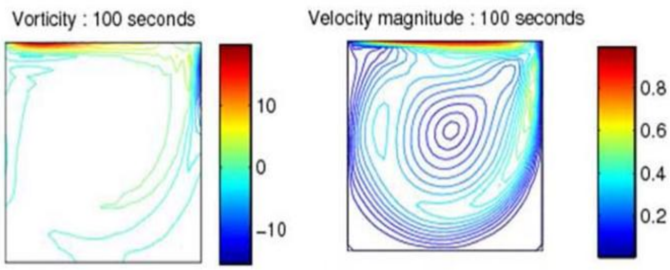

Fig.2. Vorticity (left plot), and velocity magnitude solution (right plot) using $P2 - P1$ approximation, a 64 × 64 square grid and Reynolds number Re=1000.

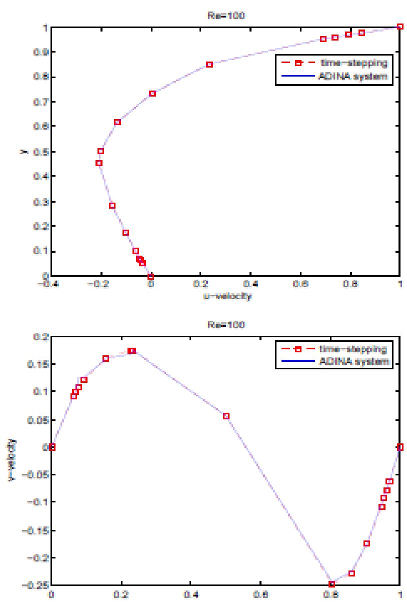

Fig.4. The velocity component u at vertical center line (left plot), and the velocity component v horizontal center line (right plot) with a 129 × 129 grid and Re=100.

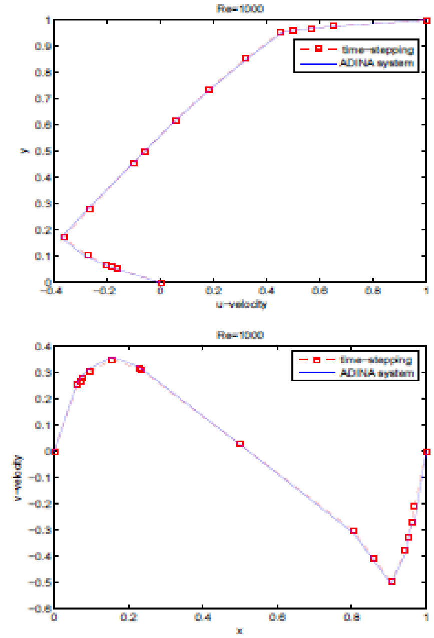

Fig.5. The velocity component u at vertical centerline (left plot), and the velocity component v at horizontal center line (right plot) with a 129 × 129 grid and Re=1000.

Figures 4 and 5 shows the velocity profiles for lines passing through the geometric center of the cavity. These features clearly demonstrate the high accuracy achieved by the proposed mixed finite element method for solving the unsteady Navier-Stokes equations in the lid-driven squared cavity.

EXAMPLE 2. L-shaped domain Ω, parabolic inflow boundary condition, natural outflow boundary condition.

This example represents flow in a rectangular duct with a sudden expansion; a Poiseuille flow profile is imposed on the inflow boundary $\{x = -1;\ 0 \le y \le 1\}$, and a no-flow (zero velocity) condition is imposed on the walls.

The Neumann condition is applied at the outflow boundary (x=5; $-1 < y < 1$) and automatically sets the mean outflow pressure to zero.

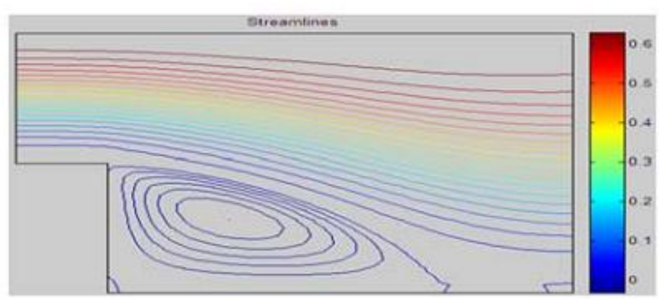

Fig.6. Equally spaced streamline plot associated with a 32 × 96 square grid, $Q_1 - P_0$ approximation and $\nu = 1/100$ (t=100 seconds).

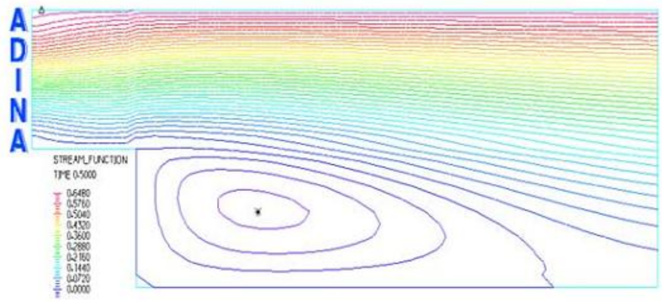

Fig.7. The solution computed with ADINA system. The plot show the streamlines associated with a 32 × 96 square grid and $\nu = 1/100$ (t=100 seconds).

The two solutions are therefore essentially identical. This is very good indication that my solver is implemented correctly.

VI. CONCLUSION

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We introduced the unsteady Navier-Stokes equations with a new boundary condition noted $C_{a,b}$. We have shown the existence and uniqueness of the solution of the weak formulation and the solution of the semi-discretization. We used the discretization by mixed finite element method.

Numerical experiments were carried out and compared with satisfaction with other numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.

ACKNOWLEDGMENTS

The authors would like to express their sincere thanks for the referee for his/her helpful suggestions.

REFERENCES

[1] Alexandre Ern, Aide-m´emoire El´ements Finis, Dunod, Paris, 2005.
[2] P.A. Raviart, J. Thomas, Introduction l'analyse numerique des équations aux d´erivées partielles, Masson, Paris, 1983.
[3] R.E. Bank, B. Welfert, A posteriori error estimates for the Stokes problem, SIAM J. Numer. Anal 28, pp. 591-623, 1991.
[4] C. Carstensen, S.A. Funken. A posteriori error control in low-order finite element discretizations of incompressible stationary flow problems. Math. Comp., 70, pp. 1353-1381, 2001.
[5] R. Verfurth, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, Chichester, 1996.
[6] D.H. Wu, I.G. Currie. Analysis of a posteriori error indicator in viscous flows. Int.J.Num. Meth.Heat Fluid Flow., 12, pp. 306-327, 2001.
[7] E. Creuse, G. Kunert, S. Nicaise, A posteriori error estimation for the Stokes problem: Anisotropic and isotropic discretizations, M3AS 14, pp. 1297-1341, 2001.
[8] P. Clement, Approximation by finite element functions using local regularization, RAIRO. Anal. Numer. 2, pp. 77-84, 1975.
[9] H. Elman, D. Silvester, A. Wathen, Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics, Oxford University Press, Oxford, 2005.
[10] M. ur Rehman, C. Vuik and G. Segal, A comparison of preconditioners for incompressible Navier-Stokes solvers, Int. J. Numer. Meth. Fluids, 57, pp.1731-1751,2007.
[11] H. Elman, V.E. Howle, J. Shadid, D. Silvester, and R. Tuminaro, Least squares preconditioners for stabilized discretizations of the Navier-Stokes equations. SIAM J. Sci. Comput., 30, 290-311, 2007.
[12] A. Gauthier, F. Saleri, and A. Veneziani, A fast preconditioner for the incompressible Navier-Stokes equations, Comput. Visual. Sci, 6, pp. 105 -112, 2004.
[13] P. Gresho, D. Gartling, J. Torczynski, K. Cliffe, K. Winters, T. Garratt, A. Spence, and J.Goodrich, Is the steady viscous incompressible 2d flow over a backward facing step at Re =800 stable?, Int. J. Numer. Methods Fluids, 17, pp. 501-541, 1993.
[14] S. Turek, A comparative study of some time-stepping techniques for the incompressible Navier-Stokes equations: From fully implicit nonlinear schemes to semi-implicit projection methods, Int. J. Numer. Meth. Fluids, 22, pp.987-1011, 1996.
[15] E. Erturk, T.C. Corke and C. Gokcol, Numerical solutions of 2-D steady incompressible driven cavity flow at high Reynolds numbers, Int. J. Numer. Meth. Fluids, 48, pp.747-774, 2005.
[16] S. Garcia, The Lid-Driven Square Cavity Flow: from Stationary to Time Periodic and Chaotic, Commun. Comput. Phys, 2, pp.900-932, 2007.
[17] E. Barragy and G.F. Carey, Stream function vorticity driven cavity solution using p finite elements, Comput. Fluids, 26, pp.453-468, 1997.
[18] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Method, Springer Verlag; New York, 1991.
[19] A. Nejat, and C. Ollivier-Gooch, Effect of discretization order on preconditioning and convergence of a high-order unstructured Newton GMRES solver for the Euler equations, J. Comput. Phys. 227, pp.2366 2386.2008.
[20] R.Temam Navier-Stokes Equations. North Holland, Amsterdam,1977.
[21] K.Yosida, Functional Analysis. Spriger-Verlag, Berlin,1965.
[22] Elman, H. and Ramage, A. and Silvester, D.J. *Algorithm 866: IFISS: a* Matlab toolbox for modelling incompressible flow. ACM Transactions on Mathematical Software, 33, pp.1-18, 2007.
[23] S.Alami1, A.Elakkad, and A. Elkhalfi, A posteriori error estimation for incompressible flow problem, IJMER, 2, pp. 533-538, 2012.
[24] Mohamed S. Ebeida, Roger L. Davis† and Roland W. Freund, Unsteady Incompressible Flow Simulation Using Galerkin Finite Elements with Spatial/Temporal Adaptation, American Institute of Aeronautics and Astronautics, pp.1-11, 2007.